

THE ARITHMETICAL RANK OF THE EDGE IDEALS OF GRAPHS WITH PAIRWISE DISJOINT CYCLES

MARGHERITA BARILE AND ANTONIO MACCHIA

ABSTRACT. We prove that, for the edge ideal of a graph whose cycles are pairwise vertex-disjoint, the arithmetical rank is bounded above by the sum of the number of cycles and the maximum height of its associated primes.

Mathematics Subject Classification (2010): 13A15, 13F55, 05C38.

Keywords: Arithmetical rank, edge ideals, cycles.

1. INTRODUCTION

Let R be a polynomial ring over a field. Any ideal I of R generated by squarefree quadratic monomials can be viewed as the so-called *edge ideal* $I(G)$ of a graph G whose vertex set $V(G)$ is the set of indeterminates, and whose edges are the sets formed by two indeterminates x, y such that xy is a generator of I . This notion was introduced in 1990 by Villarreal [22] and extensively studied in 1994 by Simis, Vasconcelos and Villarreal [21]; see [20] for a detailed survey. The present paper is concerned with two algebraic invariants of $I(G)$: the *big height*, denoted by $\text{bight } I(G)$, which is the maximum height of the minimal primes of $I(G)$, and the *arithmetical rank*, denoted by $\text{ara } I(G)$, which is the minimum number of elements of R that generate an ideal whose radical is $I(G)$. It is well known that

$$(1) \quad \text{ht } I(G) \leq \text{bight } I(G) \leq \text{pd } R/I(G) \leq \text{ara } I(G),$$

where ht and pd denote the height and the projective dimension, respectively. A very special case is the one where equality holds everywhere: then the ideal $I(G)$ is a set-theoretic complete intersection. According to some recent results, this occurs for many Cohen-Macaulay edge ideals fulfilling additional conditions like, e.g., having height two [12] or having height equal to half the number of vertices [3]. These include the bipartite graphs studied in [9] and in [6]. A more general case is the one where the arithmetical rank is equal to the projective dimension of the quotient ring. This equality has been proven for several classes of graphs, such as the graphs formed by one cycle or by two cycles having one vertex in common [2] or connected through an edge [19], those formed by some cycles and lines having a common vertex [10], or those whose edge ideals are subject to certain algebraic constraints (see, e.g., [5] and [13]). A stronger condition is the equality between the arithmetical rank and the big height, which has been established for certain unmixed bipartite graphs [15], for acyclic graphs [14], for graphs formed by a single cycle and some terminal edges attached to some of its vertices (*whisker graphs on a cycle*) (see [16] or [18]), and for graphs in which every vertex belongs to a terminal edge (*fully whiskered graphs*) (see [17] or [18]). A question that naturally arises when comparing the arithmetical rank and the big height, is whether their difference can be bounded above by means of some graph-theoretical invariants. We

will show that, for every graph whose cycles are pairwise disjoint, an upper bound is provided by the number of cycles. This is a generalization of the result in [14], but is proven independently, and by completely different techniques. The approach is inductive on the number of edges, and the basis of induction is the case of fully whiskered graphs, for which the claim was proven by the second author using the homological method, based on Lyubeznik resolutions, introduced by Kimura in [11]. All the results proven in this paper hold on any field.

2. PRELIMINARIES

We first introduce some graph-theoretical terminology and notation.

All graphs considered in this paper are simple, i.e., without multiple edges or loops. Given two vertices x and y of a graph G , we will say that x is a *neighbour* of y if the vertices x, y form an edge. By abuse of notation, this edge will be denoted by xy , with the same symbol used for the corresponding monomial of $I(G)$. The vertex x will be called *terminal* or a *leaf*, if it has exactly one neighbour y ; in this case the edge xy will be called *terminal*. For the remaining basic terminology about graphs we refer to [8].

A graph will always be identified with the set of its edges. If $G = \emptyset$, then we will set $I(G) = (0)$.

Definition 2.1. A (non-empty) graph is called a *star* if all its edges have one vertex in common.

Definition 2.2. Let G be a graph. A subset C of its vertex set is called a *vertex cover* if all edges of G have a vertex in C . A vertex cover of G is called *minimal* if it does not properly contain any vertex cover of G . A minimal vertex cover of G is called *maximum* if it has maximum cardinality among the minimal vertex covers of G .

Remark 2.3. The unique (hence, the maximum) minimal vertex cover of an empty graph is the empty set.

It is well known that the minimal vertex covers of G are the sets of generators of the minimal primes of $I(G)$. Hence $\text{bight } I(G)$ is the cardinality of the maximum minimal vertex covers of G .

Definition 2.4. Let G be a graph, and H a subgraph of G .

- (i) If $V(H) = V(G)$, we will say that G is *spanned* by H .
- (ii) If C is a minimal vertex cover of G , we will say that the (possibly empty) set $C \cap V(H)$ is the vertex cover *induced* by C on H .

Definition 2.5. Let G be a graph, and H_1 and H_2 be two subgraphs of G . If H_1 and H_2 are vertex-disjoint (i.e. have disjoint vertex sets) and there are a vertex x_1 of H_1 and a vertex x_2 of H_2 such that $G = H_1 \cup \{x_1x_2\} \cup H_2$, we will say that H_1 and H_2 are *connected through the edge* x_1x_2 .

Given a graph G , and a vertex x of G , adding a *whisker* to G at x means adding a new vertex y to the vertex set of G and the edge xy to its edge set. The new edge will be referred to as a whisker attached to x . A graph obtained from G by adding one or more whiskers to some of its vertices will be called a *whisker graph* on G . If there is at least one whisker attached to every vertex of G , G will be called *fully whiskered*. These graphs were considered by the second author in [17], were it

was proven that the arithmetical rank of their edge ideals is always equal to its big height.

In the sequel we will adopt the following notation about a graph G . If we add a whisker to G , we will denote the new graph by G' . If we remove one, two or possibly more edges from G , we will denote the resulting graph by \overline{G} , \tilde{G} , or \hat{G} , respectively. Moreover, G^\bullet and G^\vee will be alternative notations for \overline{G} and \tilde{G} , respectively. We will often use the corresponding notation for the vertex covers and the subgraphs of these graphs: for example C' will be a vertex cover of G' and H' will be a subgraph of G' .

3. MORE ON MAXIMUM MINIMAL VERTEX COVERS

In this section we will present some preliminary technical results regarding the relations between the maximum minimal vertex covers of a graph and those of certain graphs derived from it.

Lemma 3.1. *Let G be a graph and let x_1 and x_2 be two leaves of G belonging to disjoint edges. Let \dot{G} be the graph obtained from G by identifying x_1 and x_2 . Then*

$$\text{bight } I(\dot{G}) \geq \text{bight } I(G) - 1.$$

Moreover

$$\text{ara } I(\dot{G}) \leq \text{ara } I(G).$$

Proof. Let C be a maximum minimal vertex cover of G . In order to prove the first inequality, we show that \dot{G} has a minimal vertex cover of cardinality $|C|$ or $|C| - 1$. Let x be the vertex of \dot{G} obtained by identifying x_1 and x_2 . For $i = 1, 2$, let y_i be the only neighbour of x_i in G . Then $y_1 \neq y_2$.

If $x_1, x_2 \notin C$, then $y_1, y_2 \in C$. In this case, C is a minimal vertex cover of \dot{G} . So suppose that $x_1, x_2 \in C$. Then $y_1, y_2 \notin C$ and $\dot{C} = (C \setminus \{x_1, x_2\}) \cup \{x\}$ is a minimal vertex cover of \dot{G} . Finally, suppose that $x_1 \in C$, $x_2 \notin C$. In this case $y_1 \notin C$, $y_2 \in C$. If all neighbours of y_2 other than x_2 belong to C , then $\dot{C} = (C \setminus \{x_1, y_2\}) \cup \{x\}$ is a minimal vertex cover of \dot{G} . Otherwise so is $\dot{C} = (C \setminus \{x_1\}) \cup \{x\}$. This proves the first inequality. For the second inequality, let S be the set of edge monomials of $I(G)$ other than $x_1 y_1$ and $x_2 y_2$. Then the set of edge monomials of $I(\dot{G})$ is $S \cup \{x y_1, x y_2\}$. Hence, if $q_1, \dots, q_r \in R$ are such that $I(G) = \sqrt{(q_1, \dots, q_r)}$, then $I(\dot{G}) = \sqrt{(\bar{q}_1, \dots, \bar{q}_r)}$, where \bar{q} denotes the polynomial obtained from q by replacing x_1 and x_2 with x . \square

Lemma 3.2. *Let G be a graph and x one of its vertices. Let G' be the graph obtained from G by attaching a whisker to x . Then*

$$\text{bight } I(G) \leq \text{bight } I(G') \leq \text{bight } I(G) + 1.$$

Moreover, $\text{bight } I(G) = \text{bight } I(G')$ if and only if x belongs to all maximum minimal vertex covers of G (which are also maximum minimal vertex covers of G').

Proof. Let C be a maximum minimal vertex cover of G and y be the other endpoint of the whisker attached to x . If C is not a vertex cover of G' , then $x \notin C$, so that $C \cup \{y\}$ is a minimal vertex cover of G' . Hence

$$(2) \quad \text{bight } I(G) \leq \text{bight } I(G').$$

Conversely, let C' be a maximum minimal vertex cover of G' . If C' is not a minimal vertex cover of G , then $C' \setminus \{x\}$ or $C' \setminus \{y\}$ is one. Thus

$$\text{bight } I(G') \leq \text{bight } I(G) + 1.$$

Now, equality holds in (2) if and only if for all maximum minimal vertex covers C of G , $C \cup \{y\}$ is not a minimal vertex cover of G' , which, in turn, is true if and only if $x \in C$. \square

Lemma 3.3. *Let H be a subgraph of the graph G , and let x be a vertex of G such that $V(H) \cap V(G \setminus H) = \{x\}$. Let C be a minimal vertex cover of G and call D the vertex cover it induces on H .*

- (i) *If $x \notin C$, then D is minimal.*
- (ii) *If $x \in C$, then D or $D \setminus \{x\}$ is minimal.*

Proof. (i) If D is empty, then there is nothing to prove. So assume that D is not empty. Let $y \in D$. By assumption, $C \setminus \{y\}$ is not a vertex cover of G . Hence there is a neighbour z of y in G such that $z \notin C$. Note that $y \neq x$, since $x \notin D$. On the other hand, y is a vertex of H , so that $y \notin V(G \setminus H)$. Hence yz cannot be an edge of $G \setminus H$, i.e., it is an edge of H . This edge is left uncovered by $D \setminus \{y\}$, which proves the minimality of D .

(ii) Suppose that D is not minimal. Then there is $y \in D$ such that $\overline{D} = D \setminus \{y\}$ is a vertex cover of H . But $C \setminus \{y\}$ is not a vertex cover of G . Hence there is some neighbour z of y in G such that $z \notin C$. The edge yz does not belong to H , because it is not covered by \overline{D} . Hence it belongs to $G \setminus H$. Thus $y \in V(H) \cap V(G \setminus H)$, i.e., $y = x$. Next we show that \overline{D} is minimal. Let $v \in \overline{D}$. Then $v \neq x$. We prove that $\tilde{D} = \overline{D} \setminus \{v\}$ is not a vertex cover of H . By assumption $C \setminus \{v\}$ is not a vertex cover of G . Hence there is a neighbour w of v in G such that $w \notin C$. Now, if vw is an edge of H , then \tilde{D} leaves vw uncovered, which proves that \overline{D} is minimal. Otherwise vw is an edge of $G \setminus H$, but then $v \in V(H) \cap V(G \setminus H)$, which is impossible, since $v \neq x$. \square

Lemma 3.4. *Let G be a graph and let H_1 and H_2 be subgraphs of G whose vertex sets have exactly one element x in common and such that $G = H_1 \cup H_2$. Suppose that x belongs to all maximum minimal vertex covers of H_1 and of H_2 . Let D_1 and D_2 be maximum minimal vertex covers of H_1 and H_2 , respectively. Then $D_1 \cup D_2$ is a maximum minimal vertex cover of G . Moreover, x belongs to all maximum minimal vertex covers of G .*

Proof. Let $C = D_1 \cup D_2$. Then C is a vertex cover of G . We first prove that C is minimal. Let $y \in C$, say $y \in D_1$. Since D_1 is minimal, $D_1 \setminus \{y\}$ does not cover H_1 , hence there is a vertex z of H_1 such that yz is an edge of H_1 and $z \notin D_1$. But then $z \neq x$, so that $z \notin D_2$. Thus $z \notin C$. It follows that $C \setminus \{y\}$ leaves the edge yz uncovered. This proves the minimality of C .

We now prove that C is maximum. Set $d_1 = |D_1|$ and $d_2 = |D_2|$. Then $|C| = d_1 + d_2 - 1$, since x is the only common element of D_1 and D_2 . Let C^* be any minimal vertex cover of G . We show that $|C^*| \leq |C|$. Let D_1^* and D_2^* be the vertex covers induced by C^* on H_1 and H_2 , respectively. If $x \notin C^*$, then D_1^* and D_2^* are disjoint, so that $|C^*| = |D_1^*| + |D_2^*|$. Moreover, D_1^* and D_2^* are minimal on H_1 and H_2 , respectively. But, by assumption, they are not maximum, whence $|D_1^*| \leq d_1 - 1$ and $|D_2^*| \leq d_2 - 1$, so that $|C^*| \leq d_1 + d_2 - 2$. This also shows that

C^* is not maximum if $x \notin C^*$.

Now suppose that $x \in C^*$, and let $i \in \{1, 2\}$. If D_i^* is minimal, then $|D_i^*| \leq d_i$. Otherwise, by Lemma 3.3 (ii), $D_i^* \setminus \{x\}$ is a minimal vertex cover of H_i and, by assumption, it is not maximum. Hence, once again, $|D_i^*| \leq d_i$. Thus $|C^*| = |D_1^*| + |D_2^*| - 1 \leq d_1 + d_2 - 1$. \square

Lemma 3.5. *Let G be a graph formed by two graphs H_1 and H_2 connected through an edge. Let x_1 and x_2 be the endpoints of this edge, where, for $i = 1, 2$, x_i is a vertex of H_i . For $i = 1, 2$, let D_i be a maximum minimal vertex cover of H_i and set $H'_i = H_i \cup \{x_1 x_2\}$. Suppose that one of the following conditions holds:*

- (i) $x_1 \in D_1$ and, for $i = 1, 2$, x_i belongs to no maximum minimal vertex cover of H'_i ;
- (ii) x_1 belongs to all maximum minimal vertex covers of H_1 .

Then $D_1 \cup D_2$ is a maximum minimal vertex cover of G .

Proof. Set $D = D_1 \cup D_2$. Suppose that (i) or (ii) holds. Then $x_1 \in D$, so that D is a vertex cover of G . It is also minimal, because H_1 and H_2 are vertex-disjoint, and D_i is minimal on H_i , for $i = 1, 2$. We prove that it is also maximum. Let C be a minimal vertex cover of G and, for $i = 1, 2$, let E_i be the vertex cover it induces on H_i . Then E_1 and E_2 are disjoint and $C = E_1 \cup E_2$.

Suppose that (i) holds. Let $i \in \{1, 2\}$. If E_i is minimal as a vertex cover of H_i , then $|E_i| \leq |D_i|$. Otherwise, by Lemma 3.3 (ii), $x_i \in E_i$ and $E_i \setminus \{x_i\}$ is minimal, whence E_i is a minimal vertex cover of H'_i . By assumption it is not maximum, i.e., $\text{bight } I(H'_i) \geq |E_i| + 1$. Moreover, the same assumption, together with Lemma 3.2, implies that

$$\text{bight } I(H'_i) = \text{bight } I(H_i) + 1 = |D_i| + 1.$$

Hence $|E_i| \leq |D_i|$. Thus in any case we have

$$|C| = |E_1| + |E_2| \leq |D_1| + |D_2| = |D|.$$

Now suppose that (ii) holds. Then, in view of Lemma 3.2, D_1 is a maximum minimal vertex cover of H'_1 . If $x_1 \in C$, then E_1 is a minimal vertex cover of H'_1 , whence $|E_1| \leq |D_1|$. On the other hand, E_2 is a minimal vertex cover of H_2 : otherwise, by Lemma 3.3 (ii), so would be $E_2 \setminus \{x_2\}$, and thus $C \setminus \{x_2\} = E_1 \cup E_2 \setminus \{x_2\}$ would be a minimal vertex cover of G , against the minimality of C . The minimality of E_2 implies that $|E_2| \leq |D_2|$. If $x_1 \notin C$, then $x_2 \in C$ and E_1 is a non-maximum minimal vertex cover of H_1 , whereas E_2 is a minimal vertex cover of H'_2 . Hence $|E_1| \leq |D_1| - 1$ and, in view of Lemma 3.2, $|E_2| \leq |D_2| + 1$. Thus we always have $|C| \leq |D_1| + |D_2| = |D|$. \square

4. GRAPHS WITH PAIRWISE DISJOINT CYCLES

In this section we present our main result. Its proof, which will be performed by induction on the number of edges, essentially rests on the way in which the maximum minimal vertex covers of a graph G with pairwise disjoint cycles (i.e. whose cycles are pairwise vertex-disjoint) behave upon removal of special edges. Lemma 4.4 will give a complete classification of the possible cases.

Definition 4.1. Let G be a graph and x one of its vertices. A neighbour of x in G will be called *free* if it does not lie on a cycle through x .

Definition 4.2. Let G be a graph and x be one of its vertices. Let C be a minimal vertex cover of G such that $x \notin C$. Then a neighbour y of x will be called *redundant* (with respect to C) if

$$\{y\} \cup N(y) \setminus \{x\} \subset C.$$

Remark 4.3. Let C be a minimal vertex cover of G such that $x \notin C$. Then C contains a redundant neighbour y of x if and only if $C \setminus \{y\}$ is a minimal vertex cover of $G \setminus \{xy\}$. Moreover, in this case $C \cup \{x\}$ is not a minimal vertex cover of G .

Lemma 4.4. Let G be a graph with pairwise disjoint cycles, and let x be one of its vertices. Suppose that there is a maximum minimal vertex cover C of G such that $x \notin C$, and that for all minimal vertex covers with this property, there is a redundant neighbour of x . Then in C there is either a free redundant neighbour y of x or a non-free redundant neighbour z_1 of x for which the statement 1) or 2), respectively, is true.

1) Set $\overline{G} = G \setminus \{xy\}$. Then one of the following conditions holds:

- (a) $\text{bight } I(\overline{G}) = \text{bight } I(G) - 1$;
- (b) if \overline{H} is the connected component of y in \overline{G} , and $\overline{K} = \overline{G} \setminus \overline{H}$, then x belongs to all maximum minimal vertex covers of \overline{K} (so that, in particular, \overline{K} is not empty).

2) Set $\overline{G} = G \setminus \{xz_1\}$. Then call z_2 the other non-free neighbour of x and set $\tilde{G} = G \setminus \{xz_1, xz_2\}$. Then one of the following conditions holds:

- (c) $\text{bight } I(\overline{G}) = \text{bight } I(G) - 1$;
- (d) $\text{bight } I(\tilde{G}) = \text{bight } I(G) - 1$;
- (e) if \tilde{H} is the connected component of z_1 (and z_2) in \tilde{G} , and $\tilde{K} = \tilde{G} \setminus \tilde{H}$, then x belongs to all maximum minimal vertex covers of \tilde{K} (so that, in particular, \tilde{K} is not empty).

Moreover, if for all choices of C there is a free redundant neighbour of x , then (a) or (b) is true for some free redundant neighbour y of x .

Proof. Set $c = \text{bight } I(G)$. Let C be a maximum minimal vertex cover of G such that $x \notin C$. Let S be the set of redundant free neighbours of x with respect to C . We proceed by induction on $|S|$.

First suppose that $|S| = 0$. Then, according to the assumption, some non-free neighbour z_1 of x is redundant with respect to C . Then, by Remark 4.3, $C \setminus \{z_1\}$ is a minimal vertex cover of $\overline{G} = G \setminus \{xz_1\}$, so that $\text{bight } I(\overline{G}) \geq c - 1$. Suppose that (c) is not true. Then

$$(3) \quad \text{bight } I(\overline{G}) \geq \text{bight } I(G).$$

Let \overline{C} be a maximum minimal vertex cover of \overline{G} . Then \overline{C} or $\overline{C} \setminus \{x\}$ or $\overline{C} \setminus \{z_2\}$ is a minimal vertex cover of \tilde{G} , whence $\text{bight } I(\tilde{G}) \geq \text{bight } I(\overline{G}) - 1 \geq c - 1$. Suppose that also (d) is false. Then

$$(4) \quad \text{bight } I(\tilde{G}) \geq \text{bight } I(G).$$

Let D and E be the covers induced by C on \tilde{K} and \tilde{H} , respectively (see Figure 2). Note that all neighbours of x lying in \tilde{K} are free, whence, by assumption, in D there are no redundant neighbours of x . Let \tilde{C} be a maximum minimal vertex

cover of \tilde{G} . Let \tilde{D} and \tilde{E} be the covers induced by \tilde{C} on \tilde{K} and \tilde{H} , respectively. Since \tilde{K} and \tilde{H} are vertex-disjoint, D, E and \tilde{D}, \tilde{E} are disjoint. Moreover, \tilde{G} is the vertex-disjoint union of \tilde{H} and \tilde{K} , so that \tilde{C} is the disjoint union of \tilde{D} and \tilde{E} , and \tilde{D}, \tilde{E} are maximum minimal vertex covers of \tilde{K} and \tilde{H} , respectively. Thus $|\tilde{C}| = |\tilde{D}| + |\tilde{E}|$. Since G is spanned by $\tilde{H} \cup \tilde{K}$, we also have that $C = D \cup E$, and $|C| = |D| + |E|$. But, in view of (4), $|\tilde{C}| \geq |C|$. Since $x \notin C$ and x is the only vertex that \tilde{K} has in common with $G \setminus \tilde{K} = \tilde{H} \cup \{xz_1, xz_2\}$, by Lemma 3.3 (i), D is minimal. Hence $|D| \leq |\tilde{D}|$. First suppose that $x \notin \tilde{C}$. In this case the inequality $|D| < |\tilde{D}|$ would imply that $\tilde{D} \cup E$ is a minimal vertex cover of G of cardinality greater than c , which is impossible. Hence $|D| = |\tilde{D}|$, so that $|E| \leq |\tilde{E}|$. Note that, for $i = 1, 2$, if $z_i \in \tilde{E}$, then not all neighbours of z_i lying in \tilde{H} (i.e., other than x) belong to \tilde{E} . Hence one of the following cases occurs:

- $\{z_1, z_2\} \subset \tilde{E}$, in which case $D \cup \tilde{E}$ is a maximum minimal vertex cover of G without x and without redundant neighbours of x , against our assumption;
- $\{z_1, z_2\} \not\subset \tilde{E}$, in which case $D \cup \{x\} \cup \tilde{E}$ is a minimal vertex cover of G . But this, again, is impossible, since its cardinality is greater than c .

We thus conclude that $x \in \tilde{C}$ for all maximum minimal vertex covers \tilde{C} of \tilde{G} . Now, if \tilde{D}^* and \tilde{E}^* are maximum minimal vertex covers of \tilde{K} and \tilde{H} , respectively, then their union is a maximum minimal vertex cover of \tilde{G} . This shows that (e) is true, and completes the proof of the induction basis.

Now suppose that $|S| \geq 1$. Suppose that (a) is false for all choices of $y \in S$, where $\overline{G} = G \setminus \{xy\}$. Then (3) holds for all choices of $y \in S$. Let $y \in S$ be fixed. Let \overline{D} be a maximum minimal vertex cover of \overline{K} , and \overline{E} a maximum minimal vertex cover of \overline{H} (see Figure 1). Since \overline{G} is the vertex-disjoint union of \overline{H} and \overline{K} , $\overline{C} = \overline{D} \cup \overline{E}$ is a maximum minimal vertex cover of \overline{G} and $|\overline{C}| = |\overline{D}| + |\overline{E}|$.

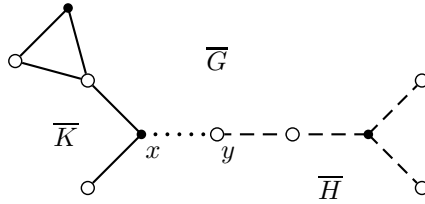


FIGURE 1.

In Figure 1 the edges of \overline{H} are dashed and the empty dots represent the vertices of the vertex cover C .

If $x \in \overline{D}$ for all choices of \overline{D} , then (b) is true.

So suppose that $x \notin \overline{D}$ for some \overline{D} . If D is the cover induced by C on \overline{K} , we then have $|\overline{D}| = |D|$, because D and \overline{D} are interchangeable in C and \overline{C} . Let E be the cover induced by C on \overline{H} . Then $|C| = |D| + |E|$. Since, in view of (3), $|\overline{C}| \geq |C|$, it follows that $|\overline{E}| \geq |E|$. If $y \in \overline{E}$ for some choice of \overline{E} , then not all neighbours of y lying in \overline{H} (i.e., other than x) belong to \overline{E} . Moreover, y is the only neighbour of x lying in \overline{H} : this follows from the fact that y is free. Hence $D \cup \overline{E}$ is a maximum minimal vertex cover of G in which the set of redundant free neighbours of x is $S \setminus \{y\}$, so that induction applies.

Now suppose that, for all y in S , we have that $x \notin \overline{D}$ for some choice of \overline{D} and $y \notin \overline{E}$ for all choices of \overline{E} . Call y_1, \dots, y_k the elements of S and, for all $i = 1, \dots, k$, let \overline{H}_i be the connected component of y_i in $\overline{G}_i = G \setminus \{xy_i\}$, and call \overline{D}_i and \overline{E}_i some maximum minimal vertex covers of $\overline{K}_i = \overline{G}_i \setminus \overline{H}_i$ and \overline{H}_i , respectively, where $x \notin \overline{D}_i$, and $y_i \notin \overline{E}_i$. Moreover, let E_i be the cover induced by C on \overline{H}_i . Note that the subgraphs \overline{H}_i are pairwise vertex-disjoint: if z were a common vertex of \overline{H}_i and \overline{H}_j , with $i \neq j$, then y_i and y_j would lie on a cycle through x and z , and would therefore not be free neighbours of x . Since $x \notin C$, E_i is also the cover induced by C on $\overline{H}_i \cup \{xy_i\}$. Recall that, for all $i = 1, \dots, k$, $|\overline{E}_i| \geq |E_i|$. Set

$$L = G \setminus \bigcup_{i=1}^k (\overline{H}_i \cup \{xy_i\}).$$

Let F be the vertex cover induced by C on L . Then $C = F \cup (\bigcup_{i=1}^k E_i)$. If in L there are no redundant neighbours of x with respect to F , then

$$F \cup \{x\} \cup \left(\bigcup_{i=1}^k \overline{E}_i \right)$$

is a minimal vertex cover of G of cardinality greater than c , which is impossible. We thus conclude that in L there is some redundant neighbour of x with respect to F (and with respect to C), which is necessarily not free. We may call it z_1 . Let \tilde{E} be the vertex cover induced by C on \tilde{H} and let \tilde{D} be the vertex cover induced by C on \tilde{K} (see Figure 2). Then $C = \tilde{D} \cup \tilde{E}$ and \tilde{D}, \tilde{E} are disjoint, so that $\tilde{D} = C \setminus \tilde{E}$.

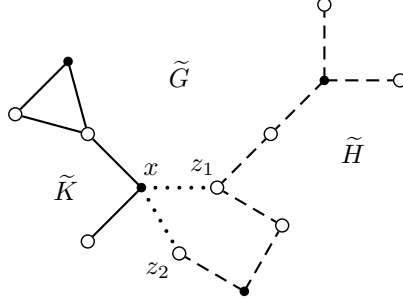


FIGURE 2.

In Figure 2 the edges of \tilde{H} are dashed and the dotted edges do not belong to \tilde{G} .

Furthermore, for all $i = 1, \dots, k$, \tilde{H} is vertex-disjoint from \overline{H}_i (and thus \tilde{E} is disjoint from E_i), because otherwise y_i would not be a free neighbour of x . Therefore

$$\tilde{D} = C \setminus \tilde{E} = (F \setminus \tilde{E}) \cup \left(\bigcup_{i=1}^k E_i \right).$$

Since $x \notin \tilde{D}$, and x is the only common vertex of \tilde{K} and $G \setminus \tilde{K} = \tilde{H} \cup \{xz_1, xz_2\}$, from Lemma 3.3 (i) it follows that \tilde{D} is a minimal vertex cover of \tilde{K} . Let \tilde{D}^* be a maximum minimal vertex cover of \tilde{K} , so that $|\tilde{D}| \leq |\tilde{D}^*|$. Suppose that $x \notin \tilde{D}^*$. In this case replacing \tilde{D} by \tilde{D}^* in $C = \tilde{D} \cup \tilde{E}$ produces a minimal vertex cover of G , and the maximality of C implies $|\tilde{D}^*| \leq |\tilde{D}|$, so that equality holds, and \tilde{D} is

maximum. Now, since $x \notin C$, we have that $z_1, z_2 \in \tilde{E}$. Thus in $F \setminus \tilde{E}$ there are no redundant neighbours of x . This implies that

$$(F \setminus \tilde{E}) \cup \{x\} \cup \left(\bigcup_{i=1}^k \overline{E}_i \right)$$

is a minimal vertex cover of \tilde{K} of cardinality greater than $|\tilde{D}|$, which contradicts the maximality of \tilde{D} . This shows that $x \in \tilde{D}^*$ for all maximum minimal vertex covers of \tilde{K} , i.e., (e) holds. \square

Corollary 4.5. (i) *If condition (b) of Lemma 4.4 holds for the graph G with respect to y , then every maximum minimal vertex cover of $\overline{G} = G \setminus \{xy\}$ contains x and is a maximum minimal vertex cover of G .*

(ii) *If condition (e) of Lemma 4.4 holds for the graph G with respect to z_1 , then every maximum minimal vertex cover of \tilde{G} contains x and is a maximum minimal vertex cover of $\overline{G} = G \setminus \{xz_1\}$.*

Proof. Suppose that condition (b) holds for G . Since \overline{G} is the vertex-disjoint union of \overline{H} and \overline{K} , the maximum minimal vertex covers of \overline{G} are the unions of a maximum minimal vertex cover of \overline{H} and a maximum minimal vertex cover of \overline{K} . This implies that all maximum minimal vertex covers of \overline{G} contain x . Moreover, since in G the subgraphs \overline{H} and \overline{K} are connected through the edge xy , the second part of claim (i) follows from Lemma 3.5 (ii).

The first part of claim (ii) follows as above from the fact that \tilde{G} is the vertex-disjoint union of \tilde{H} and \tilde{K} . For the second part, note that $\tilde{G} = \overline{G} \setminus \{xz_2\}$, and \tilde{H} and \tilde{K} are connected in \overline{G} through the edge xz_2 . Hence the claim once again follows from Lemma 3.5 (ii). \square

Lemma 4.6. *Let G be a non-empty graph with pairwise disjoint cycles. Suppose that all edges of G that do not belong to a cycle are terminal. Then every connected component of G is a star, a cycle, or a whisker graph on a cycle.*

Proof. Every connected component of G fulfils the same assumption as G . Hence it suffices to prove the claim in the case where G is connected. We first prove that G has at most one cycle. Let T_1 and T_2 be cycles of G . Let a be a vertex of T_1 and b a vertex of T_2 , where $a \neq b$. Since G is connected, in G there is a path with endpoints a and b , say $L : a = c_0 c_1 \dots c_{n-1} c_n = b$. Then, for all $i = 0, \dots, n$, c_i is not a terminal vertex, so that none of the edges of L is a terminal edge. Hence each of them must lie on a cycle. Since the cycles of G are pairwise disjoint, ac_1 must lie on T_1 . Let k be the maximum index such that the edge $c_k c_{k+1}$ of L lies on T_1 . If $k = n - 1$, then b lies on T_1 , whence $T_1 = T_2$. So assume that $k \leq n - 2$. Then $c_{k+1} c_{k+2}$ is contained in a cycle distinct from T_1 . But this is impossible, because c_{k+1} lies on T_1 and the cycles of G are pairwise disjoint. This shows that G has at most one cycle. Let H be this cycle, and suppose that $H \neq G$. Since none of the edges of $G \setminus H$ belongs to a cycle, these are all terminal edges. Since G is connected, they all have an endpoint on H . But then G is a whisker graph on a cycle. \square

Lemma 4.7. *Let G be a graph with pairwise disjoint cycles and in which there is at least one edge that is not terminal and does not belong to a cycle. Then there are two non-empty subgraphs G_1 and G_2 of G that are connected through this edge.*

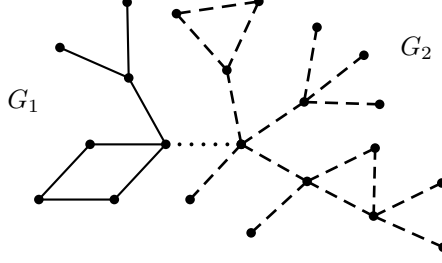


FIGURE 3.

In Figure 3 the edges of G_2 are dashed and the dotted line is the edge connecting G_1 and G_2 .

Proof. Let ax be an edge of G that is not terminal and does not belong to a cycle. Let $\overline{G} = G \setminus \{ax\}$, and let G_1 be the connected component of a in \overline{G} . Since a is not a terminal vertex, i.e., it has a neighbour other than x , G_1 is not empty. Moreover, set $G_2 = \overline{G} \setminus G_1$. Then G_1 and G_2 are vertex-disjoint, and $G = G_1 \cup \{ax\} \cup G_2$. Since ax does not belong to a cycle of G , the vertices a and x are not connected in \overline{G} , whence x is not a vertex of G_1 , i.e., it is a vertex of G_2 . Since x is not a terminal vertex, it has a neighbour $b \neq a$. But b cannot be connected to a in \overline{G} , so that b is a vertex of G_2 , i.e., bx is an edge of G_2 . This proves that G_2 is not empty, and completes the proof. \square

Theorem 4.8. *Let G be a graph with pairwise disjoint cycles. Let n be the number of its cycles. Then*

$$\text{ara } I(G) \leq \text{bight } I(G) + n.$$

Proof. Suppose that a graph G with pairwise disjoint cycles has a cycle in which at least one vertex x has degree 2. Call y_1 and y_2 the neighbours of x , which lie on the same cycle. Let L be the graph obtained by replacing the edges xy_1 and xy_2 with x_1y_1 and x_2y_2 , where x_1 and x_2 are new distinct vertices, both terminal. Then G is obtained from L by gluing together the leaves x_1 and x_2 , i.e., with respect to the notation used in Lemma 3.1, $G = \dot{L}$. Moreover, the cycles of L are still pairwise disjoint, and, if n is the number of cycles in G , the number of cycles in L is $n - 1$. Suppose that the claim of the theorem is true for L . Then it is also true for G , since, in view of Lemma 3.1,

$$\text{ara } I(G) \leq \text{ara } I(L) \leq \text{bight } I(L) + n - 1 \leq \text{bight } I(G) + n.$$

Hence, by descending induction on the number of cycles containing some vertex of degree 2, it suffices to prove the claim in the case where G has no such cycles.

Now suppose that, in addition to this condition, every edge of G that is not terminal and does not belong to a cycle has at least one whisker at each endpoint. In this case it is straightforward to verify that G is a fully whiskered graph. Then the claim is known to be true, since, according to [17], Corollary 4.2, in this case we have $\text{ara } I(G) = \text{bight } I(G)$.

Hence we may assume that

- (i) no vertex lying on a cycle of G has degree 2;

- (ii) there is an edge of G that is not terminal, does not belong to a cycle, and that has no whisker attached to one of its endpoints.

We want to prove the theorem by induction on the number of edges of G . Let ax be any edge of G that is not terminal and does not belong to a cycle, and consider the decomposition $G = G_1 \cup \{ax\} \cup G_2$ described in the proof of Lemma 4.7 (and depicted in Figure 3). We may assume that one of the endpoints of ax has no whisker attached. For all $i \in \{1, 2\}$, let $G'_i = G_i \cup \{ax\}$. Obviously, in G_1 , in G_2 , in G'_1 and G'_2 the cycles are pairwise disjoint. Hence induction applies to all these graphs. If a graph does not fulfil condition (i), it can always be reduced to a graph fulfilling (i) by means of the above construction, which leaves the number of edges unchanged. On the other hand, in view of Lemma 4.6, a graph fulfilling the above condition (i) but not (ii) is fully whiskered (this includes the case of a star, which is a whisker graph on an isolated vertex). This provides the induction basis.

In many cases considered in this proof the induction step will be performed as follows. We show that G can be decomposed as the union of two non-empty graphs A and B having exactly one vertex in common and such that for some maximum minimal vertex cover D of A and E of B , D and E are disjoint and $D \cup E$ is a minimal vertex cover of G . Then $I(G) = I(A) + I(B)$, so that $\text{ara } I(G) \leq \text{ara } I(A) + \text{ara } I(B)$, and, moreover, $\text{bight } I(G) \geq \text{bight } I(A) + \text{bight } I(B)$. On the other hand, if r and s are the numbers of cycles contained in A and B , respectively, then $n = r + s$ and, by induction, $\text{bight } I(A) \geq \text{ara } I(A) - r$ and $\text{bight } I(B) \geq \text{ara } I(B) - s$, so that

$$\text{bight } I(G) \geq \text{ara } I(A) - r + \text{ara } I(B) - s \geq \text{ara } I(G) - n,$$

whence the claim follows. In the first part of the proof, the graphs A and B will coincide with G_1 , G_2 or with G'_1 , G'_2 , or with G_1 , G'_2 .

Let C_1 and C_2 be maximum minimal vertex covers of G_1 and G_2 , respectively. Moreover, let n_1 be the number of cycles contained in G_1 and n_2 be the number of cycles contained in G_2 . Then $n = n_1 + n_2$ is the number of cycles contained in G . If $a \in C_1$ for all choices of C_1 , then by Lemma 3.5 (ii), $C_1 \cup C_2$ is a maximum minimal vertex cover of G .

Similarly, if $x \in C_2$ for all choices of C_2 , then $C_1 \cup C_2$ is a maximum minimal vertex cover of G .

Now suppose that, for some choice of C_1 and C_2 , we have $a \notin C_1$ and $x \notin C_2$. In this case, in view of Lemma 3.2, $C'_1 = C_1 \cup \{x\}$ is a maximum minimal vertex cover of G'_1 and $C'_2 = C_2 \cup \{a\}$ is a maximum minimal vertex cover of G'_2 . In particular we have

$$(5) \quad \text{bight } I(G'_1) = \text{bight } I(G_1) + 1, \quad \text{and} \quad \text{bight } I(G'_2) = \text{bight } I(G_2) + 1.$$

In the sequel, let C_1 denote any maximum minimal vertex cover of G_1 such that $a \notin C_1$ and C_2 any maximum minimal vertex cover of G_2 such that $x \notin C_2$. If, for some choice of C_1 , a has no redundant neighbours in C_1 , then $C_1 \cup C'_2$ is a minimal vertex cover of G (see Figure 4). Similarly, if, for some choice of C_2 , x has no redundant neighbours in C_2 , then $C'_1 \cup C_2$ is a minimal vertex cover of G .

So assume that for all choices of C_1 and C_2 , a and x have some redundant neighbour. Then G_1 and G_2 both fulfil the assumption of Lemma 4.4, with respect to the neighbours of a and x , respectively. Hence one of conditions (a) – (e) is true for G_1 and the same applies to G_2 .

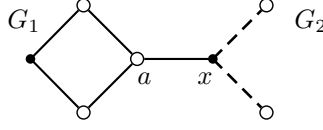


FIGURE 4.

In Figure 4 the edges of G_2 are dashed.

Let C'_1 and C'_2 be arbitrary maximum minimal vertex covers of G'_1 and G'_2 , respectively.

Now, if $a \in C'_1$ for some choice of C'_1 , then C'_1 would be a vertex cover of G_1 , but, in view of (5), not a minimal one. Then, by Lemma 3.3 (ii), $C'_1 \setminus \{a\}$ would be a minimal vertex cover of G_1 , maximum by (5). But this cover does not contain any redundant neighbours of a , because otherwise C'_1 would not be minimal. This contradicts our present assumption. Thus a does not belong to C'_1 , for all choices of C'_1 . Similarly, x does not belong to C'_2 , for all choices of C'_2 . This implies that $x \in C'_1$ for all choices of C'_1 and $a \in C'_2$ for all choices of C'_2 .

In the sequel we will use the fact that G_1 and G_2 are interchangeable, as are a and x , G'_1 and G'_2 .

Case 1 First suppose that G_2 fulfils (b) with respect to the free neighbour y of x . Set $\overline{G}_2 = G_2 \setminus \{xy\}$, let \overline{H} be the connected component of y in \overline{G}_2 and set $\overline{K} = \overline{G}_2 \setminus \overline{H}$ (see Figure 5). Consider the graph $G_1^* = G'_1 \cup \overline{K}$.

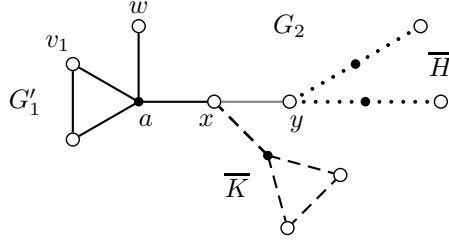


FIGURE 5.

In Figure 5 the edges of G'_1 are thick lines, the edges of \overline{H} are dotted lines, the edges of \overline{K} are dashed lines and the only edge of G_2 that does not belong to \overline{H} and \overline{K} is xy and is a thin grey line.

Let \overline{D} be a maximum minimal vertex cover of \overline{K} . In G_1^* , the subgraphs G'_1 and \overline{K} have only the vertex x in common, and, moreover, x belongs to all maximum minimal vertex covers of G'_1 and \overline{K} . Hence, by Lemma 3.4, $C_1^* = C'_1 \cup \overline{D}$ is a maximum minimal vertex cover of G_1^* and x belongs to all maximum minimal vertex covers of G_1^* . But then, according to Lemma 3.2, C_1^* is a maximum minimal vertex cover of $G_1^* \cup \{xy\}$, as well. Hence, if \overline{E} is a maximum minimal vertex cover of \overline{H} , $C_1^* \cup \overline{E}$

is a minimal vertex cover of $G = (G_1^* \cup \{xy\}) \cup \overline{H}$. Thus, if \overline{H} is not empty, the claim follows by induction applied to $G_1^* \cup \{xy\}$ and \overline{H} .

Now suppose that \overline{H} is empty. In this case xy is a terminal edge of G_2 . By Lemma 3.2, \overline{D} is a maximum minimal vertex cover of $\overline{K} \cup \{xy\}$, which, in this case, is the whole graph G_2 . Since $x \in \overline{D}$, \overline{D} is also a minimal vertex cover of G_2' . Consequently, $C_1 \cup \overline{D}$ is a minimal vertex cover of $G = G_1 \cup G_2'$, whence

$$\text{bight } I(G) \geq |C_1| + |\overline{D}|.$$

If G_1 fulfils condition (b), with respect to some free neighbour w of a , then certainly w is not a terminal vertex, because we are assuming that the edge ax is not whiskered at both endpoints. Hence, after exchanging G_1 and G_2 , we are taken back to the case in which \overline{H} is not empty. So suppose that G_1 fulfils condition (a), (c), (d) or (e). In view of Case 2 below, we only have to consider the first three cases. First suppose that G_1 fulfils (a), with respect to a redundant free neighbour w of a . Set $\overline{G}_1 = G_1 \setminus \{aw\}$ (see Figure 6). Then $\text{bight } I(\overline{G}_1) = \text{bight } I(G_1) - 1$. Moreover, let $\overline{K}' = \overline{K} \cup \{ax\}$. Then, by Lemma 3.2, \overline{D} is a maximum minimal vertex cover of \overline{K}' . Finally, set

$$\overline{G} = \overline{G}_1 \cup \overline{K}' = G \setminus \{xy, aw\}.$$

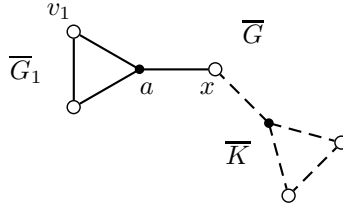


FIGURE 6.

In Figure 6 the edges of \overline{G}_1 are thick lines.

Then $I(G) = I(\overline{G}) + (xy, aw)$, whence $I(G) = \sqrt{I(\overline{G}) + (xy + aw)}$, because $ax \in I(\overline{G})$ and $x^2y^2 = xy(xy + aw) - axyw$, so that

$$\text{ara } I(G) \leq \text{ara } I(\overline{G}) + 1.$$

Now, induction applies to \overline{G}_1 and \overline{K}' , (\overline{G}_1 may be empty) so that

$$\begin{aligned} \text{bight } I(G) - 1 &\geq |C_1| - 1 + |\overline{D}| \\ &= \text{bight } I(\overline{G}_1) + \text{bight } I(\overline{K}') \\ &\geq \text{ara } I(\overline{G}_1) - n_1 + \text{ara } I(\overline{K}') - n_2 \\ &\geq \text{ara } I(\overline{G}) - n \\ &\geq \text{ara } I(G) - n - 1, \end{aligned}$$

whence the desired inequality for G .

Now suppose that (c) holds for G_1 with respect to a redundant non-free neighbour v_1 of a . Set $\overline{G}_1 = G_1 \setminus \{av_1\}$. Then $\text{bight } I(\overline{G}_1) = \text{bight } I(G_1) - 1$ and the number of cycles of \overline{G}_1 is $n_1 - 1$. Then the same computation as above yields the desired

inequality.

Finally, suppose that (d) holds for G_1 with respect to v_1 . Let v_2 be the other non-free redundant neighbour of x , set $\tilde{G}_1 = G_1 \setminus \{av_1, av_2\}$. Then $\text{bight } I(\tilde{G}_1) = \text{bight } I(G_1) - 1$ and the number of cycles of \tilde{G}_1 is $n_1 - 1$. Let

$$\tilde{G} = \tilde{G}_1 \cup \overline{K'} = G \setminus \{xy, av_1, av_2\}.$$

Then $I(G) = I(\tilde{G}) + (xy, av_1, av_2)$, whence $I(G) = \sqrt{I(\tilde{G}) + (xy + av_1, av_2)}$, and thus

$$\text{ara } I(G) \leq \text{ara } I(\tilde{G}) + 2.$$

Now, induction applies to \tilde{G}_1 and $\overline{K'}$, so that

$$\begin{aligned} \text{bight } I(G) - 1 &\geq |C_1| - 1 + |\overline{D}| \\ &= \text{bight } I(\tilde{G}_1) + \text{bight } I(\overline{K'}) \\ &\geq \text{ara } I(\tilde{G}_1) - n_1 + 1 + \text{ara } I(\overline{K'}) - n_2 \\ &\geq \text{ara } I(\tilde{G}) - n + 1 \\ &\geq \text{ara } I(G) - n - 1, \end{aligned}$$

which, again, yields the desired inequality for G .

Case 2 Now suppose that (e) applies to G_2 , with respect to some non-free neighbour z_1 of x in C_2 . Let z_2 be the other non-free neighbour of x . Set $\overline{G}_2 = G_2 \setminus \{xz_1\}$. Then induction applies to $\overline{G} = G'_1 \cup \overline{G}_2$, so that $\text{bight } I(\overline{G}) \geq \text{ara } I(\overline{G}) - n + 1$, since $n - 1$ is the number of cycles contained in \overline{G} . On the other hand, since $I(G) = I(\overline{G}) + (xz_1)$, we also have that $\text{ara } I(G) \leq \text{ara } I(\overline{G}) + 1$. Let \tilde{H} be the connected component of z_1 in $\tilde{G}_2 = G_2 \setminus \{xz_1, xz_2\}$ and set $\tilde{K} = \tilde{G}_2 \setminus \tilde{H}$. Let C'_1 be a maximum minimal vertex cover of G'_1 (whence $x \in C'_1$), \tilde{D} a maximum minimal vertex cover of \tilde{K} , and \tilde{E} a maximum minimal vertex cover of \tilde{H} . Since x is the only common vertex of G'_1 and \tilde{K} , and x belongs to all maximum minimal vertex covers of G'_1 and of \tilde{K} , by Lemma 3.4 we have that $C'_1 \cup \tilde{D}$ is a maximum minimal vertex cover of $G'_1 \cup \tilde{K}$ and x belongs to all maximum minimal vertex covers of this graph. On the other hand, $G'_1 \cup \tilde{K}$ is vertex-disjoint from \tilde{H} , and \overline{G} is obtained by connecting $G'_1 \cup \tilde{K}$ and \tilde{H} through the edge xz_2 . Hence, by Lemma 3.5 (ii), $\overline{C} = C'_1 \cup \tilde{D} \cup \tilde{E}$ is a maximum minimal vertex cover of \overline{G} . Moreover, it is a minimal vertex cover of G . Therefore $\text{bight } I(G) \geq \text{bight } I(\overline{G})$. Finally, induction applies to \overline{G} . Summing up, we have

$$\text{bight } I(G) \geq \text{bight } I(\overline{G}) \geq \text{ara } I(\overline{G}) - n + 1 \geq \text{ara } I(G) - n.$$

The cases where G_1 fulfils (b) or (e) can be treated in the same way as in Cases 1 and 2 above.

Case 3 Finally, suppose that each of G_1 and G_2 fulfils (a) or (c) or (d).

A subset W of G_1 will be called a *neighbour set* of a if $W = \{aw\}$ for some neighbour w of a or $W = \{av_1, av_2\}$ where v_1, v_2 are non-free neighbours of a . Set

$$h_1 = \max \left\{ h \left| \begin{array}{l} \exists W_1, \dots, W_h \text{ neighbour sets of } a \text{ in } G_1 \text{ such that,} \\ \text{for all } \ell = 1, \dots, h, \text{ bight } I(G_1 \setminus \bigcup_{i=1}^{\ell} W_i) = \text{bight } I(G_1) - \ell \end{array} \right. \right\}.$$

Consider

$$\widehat{G}_1 = G_1 \setminus \bigcup_{i=1}^{h_1} W_i.$$

Then $h_1 \geq 1$. In view of Lemma 4.4, the maximality of h_1 implies that either \widehat{G}_1 is empty or one of the following conditions holds.

- (i) All maximum minimal vertex covers of \widehat{G}_1 contain a (and are thus maximum minimal vertex covers of $\widehat{G}'_1 = \widehat{G}_1 \cup \{ax\}$, as well).

In the remaining cases, there is a maximum minimal vertex cover \widehat{C}_1 of \widehat{G}_1 such that $a \notin \widehat{C}_1$. In view of Lemma 3.2, this implies that $\text{bight } I(\widehat{G}'_1) = \text{bight } I(\widehat{G}_1) + 1$.

- (ii) For some maximum minimal vertex cover \widehat{C}_1 of \widehat{G}_1 such that $a \notin \widehat{C}_1$, no neighbour of a is redundant. In this case $\widehat{C}_1 \cup \{a\}$ is a maximum minimal vertex cover of \widehat{G}'_1 .

In the remaining cases, for all maximum minimal vertex covers \widehat{C}_1 of \widehat{G}_1 such that $a \notin \widehat{C}_1$, there is some redundant neighbour of a with respect to \widehat{C}_1 . This implies that a does not belong to any maximum minimal vertex cover of \widehat{G}'_1 .

- (iii) For all maximum minimal vertex covers \widehat{C}_1 of \widehat{G}_1 such that $a \notin \widehat{C}_1$, in \widehat{C}_1 there is some redundant free neighbour of a . Then, by Lemma 4.4, there is such a neighbour w for which the following holds. Set $\widehat{G}_1^\bullet = \widehat{G}_1 \setminus \{aw\}$, call \overline{H} the connected component of w in \widehat{G}_1^\bullet , and set $\overline{K} = \widehat{G}_1^\bullet \setminus \overline{H}$. Then a belongs to all maximum minimal vertex covers of \overline{K} .
- (iv) For some maximum minimal vertex cover \widehat{C}_1 of \widehat{G}_1 such that $a \notin \widehat{C}_1$, there are no redundant free neighbours of a with respect to \widehat{C}_1 , but there is a redundant non-free neighbour v_1 such that the following holds. Call v_2 the other non-free neighbour of a , and set $\widehat{G}_1^\vee = \widehat{G}_1 \setminus \{av_1, av_2\}$, call \widetilde{H} the connected component of v_1 in \widehat{G}_1^\vee , and set $\widetilde{K} = \widehat{G}_1^\vee \setminus \widetilde{H}$. Then a belongs to all maximum minimal vertex covers of \widetilde{K} .

Define h_2 for G_2 , in the same way as h_1 for G_1 , i.e., set

$$h_2 = \max \left\{ h \left| \begin{array}{l} \exists U_1, \dots, U_h \text{ neighbour sets of } x \text{ in } G_2 \text{ such that,} \\ \text{for all } \ell = 1, \dots, h, \text{ bight } I(G_2 \setminus \bigcup_{i=1}^\ell U_i) = \text{bight } I(G_2) - \ell \end{array} \right. \right\}.$$

Suppose that $h_1 \leq h_2$, and then set

$$\widehat{G}_2 = G_2 \setminus \bigcup_{i=1}^{h_1} U_i.$$

In the sequel we will admit that \widehat{G}_2 may be empty.

We have

$$I(G) = I(\widehat{G}'_1 \cup \widehat{G}_2) + I(W_1 \cup \dots \cup W_{h_1} \cup U_1 \cup \dots \cup U_{h_1}).$$

Note that at most one of the sets W_i and at most one of the sets U_i consists of two elements. For all $i = 1, \dots, h_1$, let $aw_i \in W_i$ and $xu_i \in U_i$. If some W_i contains

another element, call it α , otherwise set $\alpha = 0$. If some U_i contains another element, call it β , otherwise set $\beta = 0$. Then

$$I(G) = I(\widehat{G}'_1 \cup \widehat{G}_2) + (aw_1, \dots, aw_{h_1}, xu_1, \dots, xu_{h_1}, \alpha, \beta),$$

so that

$$I(G) = \sqrt{I(\widehat{G}'_1 \cup \widehat{G}_2) + (aw_1 + xu_1, \dots, aw_{h_1} + xu_{h_1}, \alpha + \beta)},$$

because $ax \in I(\widehat{G}'_1 \cup \widehat{G}_2)$. Hence

$$\text{ara } I(G) \leq \text{ara } I(\widehat{G}'_1 \cup \widehat{G}_2) + \widehat{h}_1 \leq \text{ara } I(\widehat{G}'_1) + \text{ara } I(\widehat{G}_2) + \widehat{h}_1,$$

where $\widehat{h}_1 = h_1$ or $\widehat{h}_1 = h_1 + 1$. In the latter case, at least one cycle is lost when passing from G to $\widehat{G}'_1 \cup \widehat{G}_2$, and consequently, if \widehat{n} is the number of cycles of this graph, we have $\widehat{n} \leq n - 1$, whereas, in general, $\widehat{n} \leq n$. Thus we always have $\widehat{n} + \widehat{h}_1 \leq n + h_1$, whence

$$h_1 - \widehat{n} \geq \widehat{h}_1 - n.$$

Moreover,

$$\text{bight } I(G_2) = \text{bight } I(\widehat{G}_2) + h_1.$$

Let \widehat{n}_1 be the number of cycles contained in \widehat{G}'_1 , and \widehat{n}_2 be the number of cycles contained in \widehat{G}_2 , so that $\widehat{n} = \widehat{n}_1 + \widehat{n}_2$. Induction applies to \widehat{G}'_1 and \widehat{G}_2 . Hence

$$\text{bight } I(\widehat{G}'_1) \geq \text{ara } I(\widehat{G}'_1) - \widehat{n}_1,$$

and

$$\text{bight } I(\widehat{G}_2) \geq \text{ara } I(\widehat{G}_2) - \widehat{n}_2.$$

In cases (i) and (ii), let \widehat{C}_1 be a maximum minimal vertex cover of \widehat{G}'_1 . Then, in case (i), $\widehat{C}'_1 = \widehat{C}_1$ is also a maximum minimal vertex cover of \widehat{G}'_1 . In case (ii), $\widehat{C}'_1 = \widehat{C}_1 \cup \{a\}$ is a maximum minimal vertex cover of \widehat{G}'_1 . In both cases, $\widehat{C}'_1 \cup C_2$ is a minimal vertex cover of G . Moreover, \widehat{C}'_1 and C_2 are disjoint. Consequently,

$$\begin{aligned} \text{bight } I(G) \geq |\widehat{C}'_1| + |C_2| &= \text{bight } I(\widehat{G}'_1) + \text{bight } I(G_2) \\ &= \text{bight } I(\widehat{G}'_1) + \text{bight } I(\widehat{G}_2) + h_1 \\ &\geq \text{ara } I(\widehat{G}'_1) - \widehat{n}_1 + \text{ara } I(\widehat{G}_2) - \widehat{n}_2 + h_1 \\ &= \text{ara } I(\widehat{G}'_1) + \text{ara } I(\widehat{G}_2) + h_1 - \widehat{n} \\ &\geq \text{ara } I(\widehat{G}'_1) + \text{ara } I(\widehat{G}_2) + \widehat{h}_1 - n \\ &\geq \text{ara } I(G) - n. \end{aligned}$$

Before we examine the remaining cases, one remark is needed. First suppose that all maximum minimal vertex covers \widehat{C}_2 of \widehat{G}_2 contain x . Then we are taken back to case (i) with \widehat{G}'_1 replaced by $\widehat{G}'_2 = \widehat{G}_2 \cup \{ax\}$ and \widehat{G}_2 replaced by \widehat{G}_1 (note that the maximality of h_1 is irrelevant in this part of the argumentation). Then suppose that for some maximum minimal vertex cover \widehat{C}_2 of \widehat{G}_2 such that $x \notin \widehat{C}_2$, there is no redundant neighbour of x with respect to \widehat{C}_2 . Then, for all neighbour sets U of x in \widehat{G}_2 , \widehat{C}_2 is also a minimal vertex cover of $\widehat{G}_2 \setminus U$. Thus the elimination of the neighbour set U does not cause the big height to drop. This implies that $h_2 = h_1$. Thus we are taken back to case (ii), with the roles of \widehat{G}'_1 and \widehat{G}_2 exchanged. Hence, in the sequel, we may assume that, for all maximum minimal vertex covers \widehat{C}_2 of \widehat{G}_2 such that $x \notin \widehat{C}_2$ (and such covers exist), there is some redundant neighbour of

x with respect to \widehat{C}_2 . Hence x does not belong to any maximum minimal vertex cover of \widehat{G}'_2 . Recall that the same is true for G'_2 .

In case (iii), let \widehat{C}_1^\bullet be a maximum minimal vertex cover of \widehat{G}_1^\bullet ; \widehat{G}_1 fulfils condition (b) with respect to the neighbour w of a , so that, by Corollary 4.5 (i), a belongs to all maximum minimal vertex covers of \widehat{G}_1^\bullet , and, in particular, $a \in \widehat{C}_1^\bullet$. Moreover, \widehat{C}_1^\bullet is a maximum minimal vertex cover of \widehat{G}_1 . Furthermore, in $\widehat{G}'_1 \cup \widehat{G}_2$, the subgraphs \widehat{G}_1 and \widehat{G}_2 are connected through the edge ax . Recall that a does not belong to any maximum minimal vertex cover of \widehat{G}'_1 . Thus, in view of Lemma 3.5 (i), $\widehat{C}_1^\bullet \cup \widehat{C}_2$ is a maximum minimal vertex cover of $\widehat{G}'_1 \cup \widehat{G}_2$. Moreover, $\widehat{C}_1^\bullet \cup C_2$ is a minimal vertex cover of G . Hence, by induction,

$$\begin{aligned} \text{bight } I(G) \geq |\widehat{C}_1^\bullet \cup C_2| &= |\widehat{C}_1^\bullet| + |C_2| = |\widehat{C}_1^\bullet| + |\widehat{C}_2| + h_1 \\ &= \text{bight } I(\widehat{G}'_1 \cup \widehat{G}_2) + h_1 \\ &\geq \text{ara } I(\widehat{G}'_1 \cup \widehat{G}_2) - \widehat{n} + h_1 \\ &\geq \text{ara } I(G) - n. \end{aligned}$$

In case (iv), let \widehat{C}_1^\vee be a maximum minimal vertex cover of \widehat{G}_1^\vee . Then, by Corollary 4.5 (ii), $a \in \widehat{C}_1^\vee$, and \widehat{C}_1^\vee is also a maximum minimal vertex cover of $\widehat{G}_1^\bullet = \widehat{G}_1 \setminus \{av_1\}$. If a does not belong to any maximum minimal vertex cover of $\widehat{G}'_1 \cup \{ax\} = \widehat{G}'_1 \setminus \{av_1\}$, then by Lemma 3.5 (i), $\widehat{C}_1^\vee \cup \widehat{C}_2$ is a maximum minimal vertex cover of $(\widehat{G}'_1 \setminus \{av_1\}) \cup \widehat{G}_2 = \widehat{G}_1^\bullet \cup \{ax\} \cup \widehat{G}_2$, since this graph is obtained by connecting \widehat{G}_1^\bullet and \widehat{G}_2 through the edge ax . Furthermore, $\widehat{C}_1^\vee \cup C_2$ is a minimal vertex cover of G .

Now, induction applies to $(\widehat{G}'_1 \setminus \{av_1\}) \cup \widehat{G}_2$, and the number of cycles contained in this graph is $\widehat{n} - 1$. Hence

$$\begin{aligned} \text{bight } I(G) \geq |\widehat{C}_1^\vee \cup C_2| &= |\widehat{C}_1^\vee \cup \widehat{C}_2| + h_1 \\ &= \text{bight } I((\widehat{G}'_1 \setminus \{av_1\}) \cup \widehat{G}_2) + h_1 \\ &\geq \text{ara } I((\widehat{G}'_1 \setminus \{av_1\}) \cup \widehat{G}_2) - \widehat{n} + 1 + h_1 \\ &\geq \text{ara } I(\widehat{G}'_1 \cup \widehat{G}_2) - 1 - \widehat{n} + 1 + h_1 \\ &\geq \text{ara } I(G) - n. \end{aligned}$$

If a belongs to some maximum minimal vertex cover \widehat{C}'_1 of $\widehat{G}'_1 \setminus \{av_1\}$, then $\widehat{C}'_1 \cup C_2$ is a minimal vertex cover of G . Moreover, $\widehat{G}'_1 \setminus \{av_1\}$ contains $n_1 - 1 = \widehat{n}_1 - 1$ cycles. Hence, by induction, we have

$$\begin{aligned} \text{bight } I(G) &\geq |\widehat{C}'_1| + |C_2| = |\widehat{C}'_1| + |\widehat{C}_2| + h_1 \\ &= \text{bight } I(\widehat{G}'_1 \setminus \{av_1\}) + \text{bight } I(\widehat{G}_2) + h_1 \\ &\geq \text{ara } I(\widehat{G}'_1 \setminus \{av_1\}) - \widehat{n}_1 + 1 + \text{ara } I(\widehat{G}_2) - \widehat{n}_2 + h_1 \\ &\geq \text{ara } I(\widehat{G}'_1) + \text{ara } I(\widehat{G}_2) - \widehat{n} + h_1 \\ &\geq \text{ara } I(G) - n. \end{aligned}$$

If $h_1 > h_2$, it suffices to apply the above arguments after exchanging the roles of G_1 and G_2 . This completes the proof. \square

5. FINAL REMARKS

According to Kuratowski's Theorem (see, e.g., [8], Theorem 11.13) the graphs whose cycles are pairwise vertex-disjoint are all planar. Moreover, the number n of cycles of a graph G fulfilling the assumption of Theorem 4.8 coincides with the so-called *cycle rank* of G , which, according to [8], Corollary 4.5 (b), is equal to $e - |V(G)| + k$, where e is the number of edges and k is the number of connected components.

Theorem 4.8 implies that, whenever G is acyclic, $\text{ara } I(G) \leq \text{bight } I(G)$. In view of (1), it follows that, in this case, $\text{ara } I(G) = \text{pd } R/I(G) = \text{bight } I(G)$. This result was conjectured in [1], where it was proven for a special class of acyclic graphs. The general case was settled by Kimura and Terai in [14]. Equality can also hold for a graph containing an arbitrary number of cycles: an infinite class of such examples is provided by the fully whiskered graphs on graphs with pairwise disjoint cycles. Therefore the inequality given in Theorem 4.8 is strict in general. On the other hand, the bound given there is sharp, as is shown by the following example, which is taken from [18].

Consider the graphs G_1 and G_2 depicted in Figure 7.

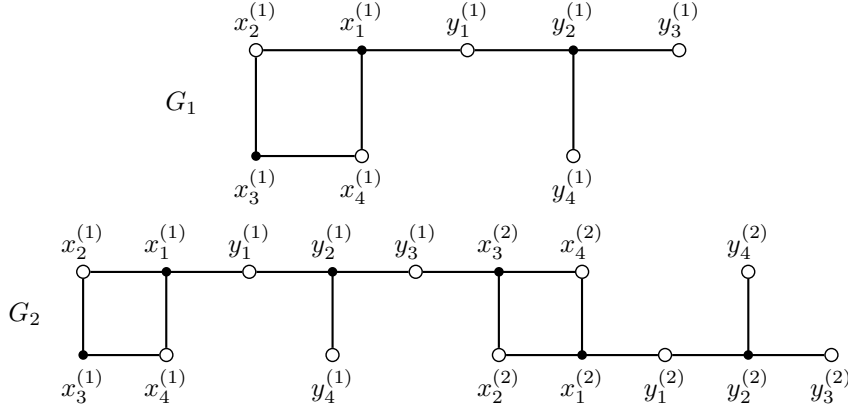


FIGURE 7.

In Figure 7 the empty dots form maximum minimal vertex covers of G_1 and G_2 .

We have $\text{bight } I(G_1) = 5$ and $\text{bight } I(G_2) = 10$. The projective dimensions of the corresponding quotient rings (in characteristic zero) can be computed by the software packages CoCoA [4] or Macaulay2 [7] and provide a lower bound for the arithmetical rank, namely $6 \leq \text{ara } I(G_1)$, and $12 \leq \text{ara } I(G_2)$. On the other hand, we also have the opposite inequalities. In fact, the polynomials

$$\begin{aligned} q_0^{(1)} &= x_1^{(1)} x_2^{(1)}, & q_3^{(1)} &= y_2^{(1)} y_3^{(1)}, \\ q_1^{(1)} &= x_1^{(1)} x_4^{(1)} + x_2^{(1)} x_3^{(1)}, & q_4^{(1)} &= y_1^{(1)} y_2^{(1)}, \\ q_2^{(1)} &= x_1^{(1)} y_1^{(1)} + x_3^{(1)} x_4^{(1)}, & q_5^{(1)} &= y_2^{(1)} y_4^{(1)} \end{aligned}$$

generate an ideal whose radical is $I(G_1)$, and the polynomials

$$\begin{aligned}
q_0^{(1)} &= x_1^{(1)} x_2^{(1)}, & q_0^{(2)} &= x_1^{(2)} x_2^{(2)}, \\
q_1^{(1)} &= x_1^{(1)} x_4^{(1)} + x_2^{(1)} x_3^{(1)}, & q_1^{(2)} &= x_1^{(2)} x_4^{(2)} + x_2^{(2)} x_3^{(2)}, \\
q_2^{(1)} &= x_1^{(1)} y_1^{(1)} + x_3^{(1)} x_4^{(1)}, & q_2^{(2)} &= x_1^{(2)} y_1^{(2)} + x_3^{(2)} x_4^{(2)}, \\
q_3^{(1)} &= y_2^{(1)} y_3^{(1)}, & q_3^{(2)} &= y_2^{(2)} y_3^{(2)}, \\
q_4^{(1)} &= y_1^{(1)} y_2^{(1)} + y_3^{(1)} x_3^{(2)}, & q_4^{(2)} &= y_1^{(2)} y_2^{(2)}, \\
q_5^{(1)} &= y_2^{(1)} y_4^{(1)}, & q_5^{(2)} &= y_2^{(2)} y_4^{(2)}
\end{aligned}$$

generate an ideal whose radical is $I(G_2)$.

Hence $6 = \text{ara } I(G_1) = \text{bight } I(G_1) + 1$ and $12 = \text{ara } I(G_2) = \text{bight } I(G_2) + 2$.

REFERENCES

- [1] M. BARILE, *On the arithmetical rank of the edge ideals of forests*, Comm. Algebra **36** (2008), 12, 4678-4703.
- [2] M. BARILE, D. KIANI, F. MOHAMMADI AND S. YASSEMI, *Arithmetical rank of the cyclic and bicyclic graphs*, J. Algebra Appl. **11** (2012), 2.
- [3] M. BARILE AND A. MACCHIA, *A note on Cohen-Macaulay graphs*. Preprint (2014). To appear in Comm. Algebra. DOI:10.1080/00927872.2015.1027361
- [4] CoCoA TEAM, *CoCoA: a system for doing Computations in Commutative Algebra*, available at <http://cocoa.dima.unige.it>.
- [5] V. ENE, O. OLTEANU AND N. TERAİ, *Arithmetical rank of lexsegment edge ideals*, Bull. Mat. Soc. Sci. Mat. Roumanie (N.S.) **53** (2010), 101, 315-327.
- [6] M. ESTRADA AND R.H. VILLARREAL, *Cohen-Macaulay bipartite graphs*, Arch. Math. **68** (1997), 124-128.
- [7] D.R. GRAYSON AND M.E. STILLMAN, *Macaulay2, a software system for research in Algebraic Geometry*, available at <http://www.math.uiuc.edu/Macaulay2/>.
- [8] F. HARARY, *Graph Theory*, Addison-Wesley, 1969.
- [9] J. HERZOG AND T. HIBI, *Distributive lattices, bipartite graphs and Alexander Duality*, J. Alg. Comb. **22** (2005), 289-302.
- [10] D. KIANI AND F. MOHAMMADI, *On the arithmetical rank of the edge ideals of some graphs*, Alg. Coll. **19** (2012), 1, 797-806.
- [11] K. KIMURA, *Lyubeznik resolutions and the arithmetical rank of monomial ideals*, Proc. Amer. Math. Soc. **137** (2009), 11, 3627-3635.
- [12] K. KIMURA, *Arithmetical rank of Cohen-Macaulay squarefree monomial ideals of height two*, J. Commutative Algebra **3** (2011), 1, 31-46.
- [13] K. KIMURA, G. RINALDO AND N. TERAİ, *Arithmetical rank of squarefree monomial ideals generated by five elements or with arithmetic degree four*, Comm. Algebra **40** (2012), 11, 4147-4170.
- [14] K. KIMURA AND N. TERAİ, *Binomial arithmetical rank of edge ideals of forests*, Proc. Amer. Math. Soc. **141** (2013), 1925-1932.
- [15] M. KUMMINI, *Regularity, depth and arithmetic rank of bipartite edge ideals*, J. Algebraic Combin. **30** (2009), 4, 429-445.
- [16] A. MACCHIA, *On the set-theoretic complete intersection property for the edge ideals of whisker graphs*, Serdica Math. J. **40** (2014), 1, 41-54.
- [17] A. MACCHIA, *The arithmetical rank of the edge ideals of graphs with whiskers*, Beitr. Algebra Geom. **56** (2015), 1, 147-158.
- [18] A. MACCHIA, *The arithmetical rank of edge ideals*. Ph.D. Thesis, University of Bari, Italy (2013).
- [19] F. MOHAMMADI, *On the edge ideals of graphs*. Ph.D. Thesis, Amirkabir University of Technology, Teheran, Iran (2005).

- [20] S. MOREY AND R.H. VILLARREAL, *Edge ideals: algebraic and combinatorial properties*. In: Progress in Commutative Algebra, Combinatorics and Homology, Vol. 1 (C. Francisco, L. C. Klingler, S. Sather-Wagstaff and J. C. Vassilev, Eds.), De Gruyter, 2012, 85-126.
- [21] A. SIMIS, W. VASCONCELOS AND R.H. VILLARREAL, *On the ideal theory of graphs*, J. Algebra **167** (1994), 2, 389-416.
- [22] R.H. VILLARREAL, *Cohen-Macaulay graphs*, Manuscripta Math. **66** (1990), 3, 277-293.

(M. Barile) DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI BARI “ALDO MORO”,
VIA ORABONA 4, 70125 BARI, ITALY

E-mail address: `margherita.barile@uniba.it`

(A. Macchia) FACHBEREICH MATHEMATIK UND INFORMATIK, PHILIPPS-UNIVERSITÄT MARBURG,
HANS-MEERWEIN-STRASSE 6, 35032 MARBURG, GERMANY

E-mail address: `macchia.antonello@gmail.com`